Exercise 25

Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$y \sin 2x = x \cos 2y, \quad (\pi/2, \pi/4)$$

Solution

The aim is to evaluate y' at $x = \pi/2$ and $y = \pi/4$ in order to find the slope there. Differentiate both sides of the given equation with respect to x.

$$\frac{d}{dx}(y \sin 2x) = \frac{d}{dx}(x \cos 2y)$$

$$\left[\frac{d}{dx}(y)\right] \sin 2x + y \left[\frac{d}{dx}(\sin 2x)\right] = \left[\frac{d}{dx}(x)\right] \cos 2y + x \left[\frac{d}{dx}(\cos 2y)\right]$$

$$(y') \sin 2x + y \left[(\cos 2x) \cdot \frac{d}{dx}(2x)\right] = (1) \cos 2y + x \left[(-\sin 2y) \cdot \frac{d}{dx}(2y)\right]$$

$$y' \sin 2x + y [(\cos 2x) \cdot (2)] = \cos 2y + x \left[(-\sin 2y) \cdot (2y')\right]$$

$$y' \sin 2x + 2y \cos 2x = \cos 2y - 2xy' \sin 2y$$

Solve for y'.

$$(\sin 2x + 2x \sin 2y)y' = \cos 2y - 2y \cos 2x$$

$$y' = \frac{\cos 2y - 2y \cos 2x}{\sin 2x + 2x \sin 2y}$$

Evaluate y' at $x = \pi/2$ and $y = \pi/4$.

$$y'\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \frac{\cos\left[2\left(\frac{\pi}{4}\right)\right] - 2\left(\frac{\pi}{4}\right) \cos\left[2\left(\frac{\pi}{2}\right)\right]}{\sin\left[2\left(\frac{\pi}{2}\right)\right] + 2\left(\frac{\pi}{2}\right) \sin\left[2\left(\frac{\pi}{4}\right)\right]} = \frac{1}{2}$$

Therefore, the equation of the tangent line to the curve represented by $y \sin 2x = x \cos 2y$ at $(\pi/2, \pi/4)$ is

$$y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right),$$

or

$$y = \frac{x}{2}.$$

Below is a graph of the curve and the tangent line at $(\pi/2, \pi/4)$.